

HOMOTOPICALLY EQUIVALENT SIMPLE LOOPS ON 2-BRIDGE SPHERES IN HECKOID ORBIFOLDS FOR 2-BRIDGE LINKS (II)

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ABSTRACT. In this paper and its prequel, we give a necessary and sufficient condition for two essential simple loops on a 2-bridge sphere in an even Heckoid orbifold for a 2-bridge link to be homotopic in the orbifold. We also give a necessary and sufficient condition for an essential simple loop on a 2-bridge sphere in an even Heckoid orbifold for a 2-bridge link to be peripheral or torsion in the orbifold. The prequel treated the case when the 2-bridge link is a $(2, p)$ -torus link, and this paper treats the remaining cases.

1. INTRODUCTION

Let $K(r)$ be the 2-bridge link of slope $r \in \mathbb{Q}$ and let n be an integer or a half-integer greater than 1. Also let $\mathcal{S}(r; n)$ be the Heckoid orbifold of index n for $K(r)$, and let $G(r; n)$ be the Heckoid group of index n for $K(r)$ which is the orbifold fundamental group of $\mathcal{S}(r; n)$. According to whether n is an integer or a non-integral half-integer, the Heckoid group $G(r; n)$ and the Heckoid orbifold $\mathcal{S}(r; n)$ are said to be *even* or *odd*.

The purpose of this paper and its prequel [6] is (i) to give a necessary and sufficient condition for two essential simple loops on a 2-bridge sphere in an even Heckoid orbifold $\mathcal{S}(r; n)$ to be homotopic in $\mathcal{S}(r; n)$, and (ii) to give a necessary and sufficient condition for an essential simple loop on a 2-bridge sphere in an even Heckoid orbifold $\mathcal{S}(r; n)$ to be peripheral or torsion in $\mathcal{S}(r; n)$. In the prequel [6], we treated the case when $K(r)$ is a $(2, p)$ -torus link, and this paper treats the remaining cases.

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Ahead of this series, the authors [3] gave a complete characterization of those essential simple loops on a 2-bridge sphere in an even Heckoid orbifold $\mathcal{S}(r; n)$ which are null-homotopic in $\mathcal{S}(r; n)$.

This paper is organized as follows. In Section 2, we describe the main results. In Section 3, we establish technical lemmas which will play essential roles in the succeeding sections. Finally, Sections 4 and 5 are devoted to the proof of Main Theorem 2.2.

2. MAIN RESULTS

This paper, as a continuation of [6], uses the same notation and terminology as in [6] without specifically mentioning. We begin with the following question about even Heckoid orbifolds, providing whose answer is the purpose of this series of papers.

Question 2.1. For r a rational number and n an integer greater than 1, consider the even Heckoid orbifold $\mathcal{S}(r; n)$ of index n for the 2-bridge link $K(r)$.

- (1) For two distinct essential simple loops α_s and $\alpha_{s'}$ on \mathcal{S} , when are they homotopic in $\mathcal{S}(r; n)$?
- (2) Which essential simple loop α_s on \mathcal{S} is peripheral or torsion in $\mathcal{S}(r; n)$?

In the prequel [6], we treated the case when $r = 1/p$ for some integer $p \geq 2$, and obtained a complete answer (see [6, Main Theorem 2.5]). In the present paper, we solve the above question for the remaining cases.

Main Theorem 2.2. *Suppose that r is a non-integral rational number and that n is an integer greater than 1. Then the following hold.*

- (1) *The simple loops $\{\alpha_s \mid s \in I(r; n)\}$ represent mutually distinct conjugacy classes in $G(r; n)$.*
- (2) *There is no rational number $s \in I(r; n)$ for which α_s is peripheral in $G(r; n)$.*
- (3) *There is no rational number $s \in I(r; n)$ for which α_s is torsion in $G(r; n)$.*

The key tool used in the proofs is small cancellation theory applied to the upper presentations of even Heckoid groups.

Remark 2.3. (1) When r is an integer, the Heckoid group $G(r; n) \cong G(0; n)$ is isomorphic to the subgroup $\langle P, SPS^{-1} \rangle$ of the classical Hecke group $\langle P, S \rangle$

introduced in [1], where

$$P = \begin{pmatrix} 1 & 2 \cos \frac{\pi}{2n} \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is plausible that Main Theorem 2.2 is also valid even when r is an integer. However, we cannot directly apply the arguments of this paper, and this case will be treated elsewhere.

(2) By Schubert's classification of 2-bridge links together with (1), we may assume that r is a rational number with $0 < r \leq 1/2$. Since the case when $r = 1/p$ for some integer $p \geq 2$ was already treated in [6], it remains to prove Main Theorem 2.2 for a rational number r with $0 < r < 1/2$ such that $r \neq 1/p$ for any integer $p \geq 2$.

(3) Since Main Theorem 2.2(2) can be proved by simply replacing $1/p$ with a non-integral rational number r in [6, Section 7], we skip its proof in the present paper.

(4) Main Theorem 2.2(1) together with Main Theorem 2.2(3) implies that the simple loops $\{\alpha_s \mid s \in I(r; n) \cup \{r\}\}$ are not mutually homotopic in $\mathbf{S}(r; n)$, since α_r is clearly torsion in $\mathbf{S}(r; n)$.

3. TECHNICAL LEMMAS

In the remainder of this paper unless specified otherwise, suppose that r is a rational number with $0 < r < 1$ such that $r \neq 1/p$ for any integer $p \geq 2$, and let n be an integer with $n \geq 2$. Write r as a continued fraction expansion $r = [m_1, m_2, \dots, m_k]$, where $k \geq 2$, $(m_1, \dots, m_k) \in (\mathbb{Z}_+)^k$ and $m_k \geq 2$. Recall that the region, R , bounded by a pair of Farey edges with an endpoint ∞ and a pair of Farey edges with an endpoint r forms a fundamental domain for the action of $\Gamma(r; n)$ on \mathbb{H}^2 (see [6, Figure 1]). Let $I_1(r; n)$ and $I_2(r; n)$ be the (closed or half-closed) intervals in \mathbb{R} defined as follows:

$$I_1(r; n) = \begin{cases} [0, r_1], & \text{where } r_1 = [m_1, \dots, m_{k-1}, m_k - 1, 2], & \text{if } k \text{ is even,} \\ [0, r_1), & \text{where } r_1 = [m_1, \dots, m_k, 2n - 2], & \text{if } k \text{ is odd,} \end{cases}$$

$$I_2(r; n) = \begin{cases} (r_2, 1], & \text{where } r_2 = [m_1, \dots, m_k, 2n - 2], & \text{if } k \text{ is even,} \\ [r_2, 1], & \text{where } r_2 = [m_1, \dots, m_{k-1}, m_k - 1, 2], & \text{if } k \text{ is odd.} \end{cases}$$

Then we may choose a fundamental domain R so that the intersection of \bar{R} with $\partial\mathbb{H}^2$ is equal to the union $\bar{I}_1(r; n) \cup \bar{I}_2(r; n) \cup \{\infty, r\}$.

3.1. The case when $s \in I_1(r; n) \cup I_2(r; n)$

In this subsection, we investigate important properties of $CS(s)$ for a rational number s such that $s \in I_1(r; n) \cup I_2(r; n)$. These properties will be used in the proof of Main Theorem 2.2 in the succeeding sections. The following lemma is a slight refinement of [3, Proposition 5.1].

Lemma 3.1. *Let $S(r) = (S_1, S_2, S_1, S_2)$ be as in [6, Lemma 3.9]. Then for any $s \in I_1(r; n) \cup I_2(r; n)$, the following hold.*

- (1) *If k is even, then $CS(s)$ does not contain $((2n - 2)\langle S_1, S_2 \rangle, S_1)$ as a subsequence.*
- (2) *If k is odd, then $CS(s)$ does not contain $((2n - 2)\langle S_2, S_1 \rangle, S_2)$ as a subsequence.*

Proof. We prove (1) and (2) simultaneously by induction on $k \geq 2$. For simplicity, we write m for m_1 . By [6, Lemma 3.9], S_1 begins and ends with $m + 1$, and S_2 begins and ends with m . Suppose on the contrary that there exists some $s \in I_1(r; n) \cup I_2(r; n)$ for which $CS(s)$ contains $((2n - 2)\langle S_1, S_2 \rangle, S_1)$ as a subsequence provided k is even and $((2n - 2)\langle S_2, S_1 \rangle, S_2)$ as a subsequence provided k is odd. This implies by [6, Lemma 3.5] that $CS(s)$ consists of m and $m + 1$. So $s \neq 0$ and s has a continued fraction expansion $s = [l_1, \dots, l_t]$, where $t \geq 2$, $(l_1, \dots, l_t) \in (\mathbb{Z}_+)^t$, $l_1 = m$ and $l_t \geq 2$. For the rational numbers r and s , define the rational numbers \tilde{r} and \tilde{s} as in [6, Lemma 3.8] so that $CS(\tilde{r}) = CT(r)$ and $CS(\tilde{s}) = CT(s)$.

We consider three cases separately.

Case 1. $m_2 = 1$.

In this case, $k \geq 3$ and, by [6, Corollary 3.14(1)], $(m + 1, m + 1)$ appears in S_1 , so in $CS(s)$, as a subsequence. Thus by [6, Lemma 3.5], $l_2 = 1$ and so $t \geq 3$. So, we have

$$\tilde{r} = [m_3, \dots, m_k] \quad \text{and} \quad \tilde{s} = [l_3, \dots, l_t].$$

It follows from $s \in I_1(r; n) \cup I_2(r; n)$ that $\tilde{s} \in I_1(\tilde{r}; n) \cup I_2(\tilde{r}; n)$. At this point, we divide this case into three subcases.

Case 1.a. $k = 3$.

By [6, Lemma 3.12(1)], $S_1 = (m_3 \langle m + 1 \rangle)$ and $S_2 = (m)$. Since $((2n - 2)\langle S_2, S_1 \rangle, S_2)$ is contained in $CS(s)$ by the assumption, this implies that $CS(\tilde{s}) = CT(s)$ contains $((2n - 2)\langle m_3 \rangle)$ as a subsequence. But since $\tilde{r} = 1/m_3 = [m_3]$ and $\tilde{s} \in I_1(\tilde{r}; n) \cup I_2(\tilde{r}; n)$, this gives a contradiction to [6, Lemma 5.1].

Case 1.b. $k \geq 4$ and k is even.

Let $S(\tilde{r}) = (T_1, T_2, T_1, T_2)$ be the decomposition of $S(\tilde{r})$ given by [6, Lemma 3.9]. Since S_1 begins and ends with $m + 1$, S_2 begins and ends with m , and since $((2n - 2)\langle S_1, S_2 \rangle, S_1)$ is contained in $CS(s)$ by the assumption, we see by [6, Lemma 3.12(2)] that $CS(\tilde{s}) = CT(s)$ contains, as a subsequence,

$$(t_1 + \ell', t_2, \dots, t_{s_1-1}, t_{s_1}, T_2, (2n - 3)\langle T_1, T_2 \rangle, t_1, t_2, \dots, t_{s_1-1}, t_{s_1} + \ell''),$$

where $(t_1, t_2, \dots, t_{s_1}) = T_1$ and $\ell', \ell'' \in \mathbb{Z}_+ \cup \{0\}$. Since $t_1 = t_{s_1} = m_3 + 1$ by [6, Lemma 3.9], this actually implies that $\ell' = \ell'' = 0$, and therefore $CS(\tilde{s})$ contains $((2n - 2)\langle T_1, T_2 \rangle, T_1)$ as a subsequence. But since $\tilde{r} = [m_3, \dots, m_k]$ and $\tilde{s} \in I_1(\tilde{r}; n) \cup I_2(\tilde{r}; n)$, this gives a contradiction to the induction hypothesis.

Case 1.c. $k \geq 4$ and k is odd.

Let $S(\tilde{r}) = (T_1, T_2, T_1, T_2)$ be the decomposition of $S(\tilde{r})$ given by [6, Lemma 3.9]. Since S_1 begins and ends with $m + 1$, S_2 begins and ends with m , and since $((2n - 2)\langle S_2, S_1 \rangle, S_2)$ is contained in $CS(s)$ by the assumption, we see by [6, Lemma 3.12(2)] that $CS(\tilde{s}) = CT(s)$ contains $((2n - 2)\langle T_2, T_1 \rangle, T_2)$ as a subsequence. But since $\tilde{r} = [m_3, \dots, m_k]$ and $\tilde{s} \in I_1(\tilde{r}; n) \cup I_2(\tilde{r}; n)$, this gives a contradiction to the induction hypothesis.

Case 2. $k = 2$ and $m_2 = 2$.

In this case, $r = [m, 2]$, so by [6, Lemma 3.12(3)], $S_1 = (m + 1)$ and $S_2 = (m)$. Since $((2n - 2)\langle S_1, S_2 \rangle, S_1)$ is contained in $CS(s)$ by the assumption, $((2n - 2)\langle m + 1, m \rangle, m + 1)$ is contained in $CS(s)$. This implies that $CS(\tilde{s}) = CT(s)$ contains $((2n - 2)\langle 1 \rangle)$ as a subsequence. Then the argument in [3, Case 2 in Proof of Proposition 5.1(1)] implies that this case cannot happen.

Case 3. Either both $k = 2$ and $m_2 \geq 3$ or both $k \geq 3$ and $m_2 \geq 2$.

In this case, by [6, Corollary 3.14(2)], (m, m) appears in S_2 , so in $CS(s)$, as a subsequence. So $l_2 \geq 2$ by [6, Lemma 3.5], and thus we have

$$\tilde{r} = [m_2 - 1, m_3, \dots, m_k] \quad \text{and} \quad \tilde{s} = [l_2 - 1, l_3, \dots, l_t].$$

It follows from $s \in I_1(r; n) \cup I_2(r; n)$ that $\tilde{s} \in I_1(\tilde{r}; n) \cup I_2(\tilde{r}; n)$. At this point, we consider three subcases separately.

Case 3.a. $k = 2$ and $m_2 \geq 3$.

By [6, Lemma 3.12(3)], $S_1 = (m + 1)$ and $S_2 = ((m_2 - 1)\langle m \rangle)$. Since $((2n - 2)\langle S_1, S_2 \rangle, S_1)$ is contained in $CS(s)$ by the assumption, $CS(\tilde{s}) = CT(s)$ contains $((2n - 2)\langle m_2 - 1 \rangle)$ as a subsequence. But since $\tilde{r} = 1/(m_2 - 1) = [m_2 - 1]$ and $\tilde{s} \in I_1(\tilde{r}; n) \cup I_2(\tilde{r}; n)$, this gives a contradiction to [6, Lemma 5.1].

Case 3.b. $k \geq 3$ is even and $m_2 \geq 2$.

Let $S(\tilde{r}) = (T_1, T_2, T_1, T_2)$ be the decomposition of $S(\tilde{r})$ given by [6, Lemma 3.9]. Since S_1 begins and ends with $m + 1$, S_2 begins and ends with m , and since $((2n - 2)\langle S_1, S_2 \rangle, S_1)$ is contained in $CS(s)$ by the assumption, we see by [6, Lemma 3.12(4)] that $CS(\tilde{s}) = CT(s)$ contains $((2n - 2)\langle T_2, T_1 \rangle, T_2)$ as a subsequence. But since $\tilde{r} = [m_2 - 1, m_3, \dots, m_k]$ and $\tilde{s} \in I_1(\tilde{r}; n) \cup I_2(\tilde{r}; n)$, this gives a contradiction to the induction hypothesis.

Case 3.c. $k \geq 3$ is odd and $m_2 \geq 2$.

Let $S(\tilde{r}) = (T_1, T_2, T_1, T_2)$ be the decomposition of $S(\tilde{r})$ given by [6, Lemma 3.9]. Since S_1 begins and ends with $m + 1$, S_2 begins and ends with m , and since $((2n - 2)\langle S_2, S_1 \rangle, S_2)$ is contained in $CS(s)$ by the assumption, we see by [6, Lemma 3.12(4)] that $CS(\tilde{s}) = CT(s)$ contains, as a subsequence,

$$(t_1 + \ell', t_2, \dots, t_{s_1-1}, t_{s_1}, T_2, (2n - 3)\langle T_1, T_2 \rangle, t_{s_1-1}, t_{s_1} + \ell''),$$

where $(t_1, t_2, \dots, t_{s_1}) = T_1$ and $\ell', \ell'' \in \mathbb{Z}_+ \cup \{0\}$. Since $t_1 = t_{s_1} = (m_2 - 1) + 1 = m_2$ by [6, Lemma 3.9], this actually implies that $\ell' = \ell'' = 0$, and therefore $CS(\tilde{s})$ contains $((2n - 2)\langle T_1, T_2 \rangle, T_1)$ as a subsequence. But since $\tilde{r} = [m_2 - 1, m_3, \dots, m_k]$ and $\tilde{s} \in I_1(\tilde{r}; n) \cup I_2(\tilde{r}; n)$, this gives a contradiction to the induction hypothesis.

The proof of Lemma 3.1 is now completed. \square

As an easy consequence of Lemma 3.1 and [6, Lemma 4.3], we obtain the following.

Corollary 3.2. *For any $s \in I_1(r; n) \cup I_2(r; n)$, the cyclic word (u_s) cannot contain a subword w of the cyclic word $(u_r^{\pm n})$ which is a product of $4n - 1$ pieces but is not a product of less than $4n - 1$ pieces.*

3.2. The case when $s \in I_1(r) \cup I_2(r)$

If Γ_r is the group of automorphisms of the Farey tessellation \mathcal{D} generated by reflections in the edges of \mathcal{D} with an endpoint r , and $\hat{\Gamma}_r$ is the group generated by Γ_r and Γ_∞ , then the region, Q , bounded by a pair of Farey edges with an endpoint ∞ and a pair of Farey edges with an endpoint r forms a fundamental domain of the action of $\hat{\Gamma}_r$ on \mathbb{H}^2 . Let $I_1(r)$ and $I_2(r)$ be the closed intervals in $\hat{\mathbb{R}}$ obtained as the intersection with $\hat{\mathbb{R}}$ of the closure of Q . Then the intervals

$I_1(r)$ and $I_2(r)$ are given by $I_1(r) = [0, \hat{r}_1]$ and $I_2(r) = [\hat{r}_2, 1]$, where

$$\hat{r}_1 = \begin{cases} [m_1, m_2, \dots, m_{k-1}, m_k - 1] & \text{if } k \text{ is even,} \\ [m_1, m_2, \dots, m_{k-1}] & \text{if } k \text{ is odd,} \end{cases}$$

$$\hat{r}_2 = \begin{cases} [m_1, m_2, \dots, m_{k-1}] & \text{if } k \text{ is even,} \\ [m_1, m_2, \dots, m_{k-1}, m_k - 1] & \text{if } k \text{ is odd.} \end{cases}$$

Clearly $I_1(r) \subsetneq I_1(r; n)$ and $I_2(r) \subsetneq I_2(r; n)$. It was shown in [7, Proposition 4.6] that if two elements s and s' of $\hat{\mathbb{Q}}$ belong to the same $\hat{\Gamma}_r$ -orbit, then the unoriented loops α_s and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$.

Lemma 3.3. *Let $S(r) = (S_1, S_2, S_1, S_2)$ be as in [6, Lemma 3.9]. For any $s \in I_1(r) \cup I_2(r)$, either S_1 or S_2 cannot occur in $CS(s)$ as a subsequence.*

Proof. The assertion for the case when $s \neq 0$ are nothing other than [4, Proposition 3.19], while the assertion for the case $s = 0$ follows from the fact that $CS(u_0) = ((2))$ (see [6, Remark 3.2]). \square

3.3. The case when $s \in I_1(r; n) \setminus I_1(r)$ provided k is even, and $s \in I_2(r; n) \setminus I_2(r)$ provided k is odd

In this subsection, we investigate an important property of $CS(s)$ for a rational number s such that

$$\begin{cases} s \in I_1(r; n) \setminus I_1(r) & \text{if } k \text{ is even;} \\ s \in I_2(r; n) \setminus I_2(r) & \text{if } k \text{ is odd.} \end{cases}$$

Lemma 3.4. *Let $S(r) = (S_1, S_2, S_1, S_2)$ be as in [6, Lemma 3.9]. For simplicity, we write m for m_1 .*

- (1) *If k is even and $[m_1, \dots, m_k - 1] < s \leq [m_1, \dots, m_k - 1, 2]$, then $CS(s)$ contains $(m+1, S_{2e}, S_1, S_2, S_1, S_{2b}, m+1)$ as a subsequence, where $(m, S_{2e}) = (S_{2b}, m) = S_2$.*
- (2) *If k is odd and $[m_1, \dots, m_k - 1, 2] \leq s < [m_1, \dots, m_k - 1]$, then $CS(s)$ contains $(m, S_{1e}, S_2, S_1, S_2, S_{1b}, m)$ as a subsequence, where $(m+1, S_{1e}) = (S_{1b}, m+1) = S_1$.*

Proof. We prove (1) and (2) simultaneously by induction on $k \geq 2$. Let s satisfy

$$\begin{cases} [m_1, \dots, m_k - 1] < s \leq [m_1, \dots, m_k - 1, 2] & \text{if } k \text{ is even;} \\ [m_1, \dots, m_k - 1, 2] \leq s < [m_1, \dots, m_k - 1] & \text{if } k \text{ is odd.} \end{cases}$$

Write s as a continued fraction expansion $s = [l_1, \dots, l_t]$, where $t \geq 1$, $(l_1, \dots, l_t) \in (\mathbb{Z}_+)^t$ and $l_t \geq 2$. Then we have $t \geq k+1$, $l_1 = m_1, \dots, l_{k-1} = m_{k-1}$, $l_k = m_k - 1$ and $l_{k+1} \geq 2$.

Throughout the proof, denote by \tilde{r} and \tilde{s} the rational numbers defined as in [6, Lemma 3.8] for the rational numbers r and s , so that $CS(\tilde{r}) = CT(r)$ and $CS(\tilde{s}) = CT(s)$.

We consider three cases separately.

Case 1. $m_2 = 1$.

In this case, $k \geq 3$, $l_2 = m_2 = 1$ and $t \geq 4$. So we have

$$\tilde{r} = [m_3, \dots, m_k] \quad \text{and} \quad \tilde{s} = [l_3, \dots, l_t].$$

It follows from the assumption that

$$\begin{cases} [m_3, \dots, m_k - 1] < \tilde{s} \leq [m_3, \dots, m_k - 1, 2] & \text{if } k \text{ is even;} \\ [m_3, \dots, m_k - 1, 2] \leq \tilde{s} < [m_3, \dots, m_k - 1] & \text{if } k \text{ is odd.} \end{cases}$$

This enables us to use the inductive argument. At this point, we divide this case into three subcases.

Case 1.a. $k = 3$.

Since $[m_3 - 1, 2] \leq \tilde{s} < [m_3 - 1]$, we see by [6, Lemma 5.5] that $CS(\tilde{s})$ contains $(m_3 - 1, m_3, m_3 - 1)$ as a subsequence. Since $CT(s) = CS(\tilde{s})$, this implies that $CS(s)$ contains a subsequence

$$(m, (m_3 - 1)\langle m + 1 \rangle, m, m_3\langle m + 1 \rangle, m, (m_3 - 1)\langle m + 1 \rangle, m).$$

Since $S_1 = (m_3\langle m + 1 \rangle)$ and $S_2 = (m)$ by [6, Lemma 3.12(1)], $CS(s)$ contains $(m, S_{1e}, S_2, S_1, S_2, S_{1b}, m)$ as a subsequence, where $(m+1, S_{1e}) = (S_{1b}, m+1) = S_1$. So the assertion holds.

Case 1.b. $k \geq 4$ is even.

Let $S(\tilde{r}) = (T_1, T_2, T_1, T_2)$ be the decomposition of $S(\tilde{r})$ given by [6, Lemma 3.9]. Since $[m_3, \dots, m_k - 1] < \tilde{s} \leq [m_3, \dots, m_k - 1, 2]$, by the induction hypothesis $CS(\tilde{s})$ contains $(m_3 + 1, T_{2e}, T_1, T_2, T_1, T_{2b}, m_3 + 1)$ as a subsequence, where $(m_3, T_{2e}) = (T_{2b}, m_3) = T_2$. Since $CS(\tilde{s}) = CT(s)$, we see by using [6, Lemma 3.12(2)] that $CS(s)$ contains $(m + 1, S_{2e}, S_1, S_2, S_1, S_{2b}, m + 1)$ as a subsequence, where $(m, S_{2e}) = (S_{2b}, m) = S_2$.

Case 1.c. $k \geq 4$ is odd.

Let $S(\tilde{r}) = (T_1, T_2, T_1, T_2)$ be the decomposition of $S(\tilde{r})$ given by [6, Lemma 3.9]. Then, by the induction hypothesis, $CS(\tilde{s})$ contains $(m_3, T_{1e}, T_2, T_1, T_2, T_{1b}, m_3)$

as a subsequence, where $(m_3 + 1, T_{1e}) = (T_{1b}, m_3 + 1) = T_1$. Since $CS(\tilde{s}) = CT(s)$, we see by using [6, Lemma 3.12(2)] that $CS(s)$ contains $(m, S_{1e}, S_2, S_1, S_2, S_{1b}, m)$ as a subsequence, where $(m + 1, S_{1e}) = (S_{1b}, m + 1) = S_1$.

Case 2. $k = 2$ and $m_2 = 2$.

In this case, $r = [m, 2]$ and $[m + 1] < s \leq [m, 1, 2]$. Then for $s = [l_1, \dots, l_t]$, we have $t \geq 3$, $l_1 = m$, $l_2 = 1$ and $l_3 \geq 2$, so that $\tilde{s} = [l_3, \dots, l_t]$ with $t \geq 3$ and $l_3 \geq 2$. Hence $CS(\tilde{s}) = CT(s)$ contains (l_3, l_3) or $(l_3, l_3 + 1)$ as a subsequence. Since $l_3 \geq 2$, this implies that $CS(s)$ contains $(m + 1, m + 1, m, m + 1, m + 1)$ as a subsequence. Since $S_1 = (m + 1)$ and $S_2 = (m)$ by [6, Lemma 3.12(3)] and hence $S_{2e} = S_{2b} = \emptyset$, $CS(s)$ contains a subsequence $(m + 1, S_{2e}, S_1, S_2, S_1, S_{2b}, m + 1)$, so the assertion hold.

Case 3. Either both $k = 2$ and $m_2 \geq 3$ or both $k \geq 3$ and $m_2 \geq 2$.

In this case, $l_2 \geq 2$. So we have

$$\tilde{r} = [m_2 - 1, \dots, m_k] \quad \text{and} \quad \tilde{s} = [l_2 - 1, \dots, l_t].$$

It follows from the assumption that

$$\begin{cases} [m_2 - 1, m_3, \dots, m_k - 1, 2] \leq \tilde{s} < [m_2 - 1, m_3, \dots, m_k - 1] & \text{if } k \text{ is even;} \\ [m_2 - 1, m_3, \dots, m_k - 1] < \tilde{s} \leq [m_2 - 1, m_3, \dots, m_k - 1, 2] & \text{if } k \text{ is odd.} \end{cases}$$

This enables us to use the inductive argument. At this point, we divide this case into three subcases.

Case 3.a. $k = 2$ and $m_2 \geq 3$.

In this case, $\tilde{r} = [m_2 - 1]$. Since $[m_2 - 2, 2] \leq \tilde{s} < [m_2 - 2]$, we see by [6, Lemma 5.5] that $CS(\tilde{s})$ contains $(m_2 - 2, m_2 - 1, m_2 - 2)$ as a subsequence. Since $CT(s) = CS(\tilde{s})$, this implies that $CS(s)$ contains

$$(m + 1, (m_2 - 2)\langle m \rangle, m + 1, (m_2 - 1)\langle m \rangle, m + 1, (m_2 - 2)\langle m \rangle, m + 1),$$

as a subsequence. Since $S_1 = (m + 1)$ and $S_2 = ((m_2 - 1)\langle m \rangle)$ by [6, Lemma 3.12(3)], $CS(s)$ contains a subsequence $(m + 1, S_{2e}, S_1, S_2, S_1, S_{2b}, m + 1)$, where $(m, S_{2e}) = (S_{2b}, m) = S_2$. Hence the assertion holds.

Case 3.b. $k \geq 3$ is even and $m_2 \geq 2$.

Let $S(\tilde{r}) = (T_1, T_2, T_1, T_2)$ be the decomposition of $S(\tilde{r})$ given by [6, Lemma 3.9]. Then, by the induction hypothesis, $CS(\tilde{s})$ contains $(m_2 - 1, T_{1e}, T_2, T_1, T_2, T_{1b}, m_2 - 1)$ as a subsequence, where $(m_2, T_{1e}) = (T_{1b}, m_2) = T_1$. Since $CS(\tilde{s}) = CT(s)$, we see by using [6, Lemma 3.12(4)] that $CS(s)$ contains $(m + 1, S_{2e}, S_1, S_2, S_1, S_{2b}, m + 1)$ as a subsequence, where $(m, S_{2e}) = (S_{2b}, m) = S_2$.

Case 3.c. $k \geq 3$ is odd and $m_2 \geq 2$.

Let $S(\tilde{r}) = (T_1, T_2, T_1, T_2)$ be the decomposition of $S(\tilde{r})$ given by [6, Lemma 3.9]. Then, by the induction hypothesis, $CS(\tilde{s})$ contains $(m_2, T_{2e}, T_1, T_2, T_1, T_{2b}, m_2)$ as a subsequence, where $(m_2 - 1, T_{2e}) = (T_{2b}, m_2 - 1) = T_2$. Since $CS(\tilde{s}) = CT(s)$, we see by using [6, Lemma 3.12(4)] that $CS(s)$ contains $(m, S_{1e}, S_2, S_1, S_2, S_{1b}, m)$ as a subsequence, where $(m + 1, S_{1e}) = (S_{1b}, m + 1) = S_1$.

The proof of Lemma 3.4 is now completed. \square

3.4. The case when $s \in I_2(r; n) \setminus I_2(r)$ provided k is even, and $s \in I_1(r; n) \setminus I_1(r)$ provided k is odd

Finally, we investigate an important property of $CS(s)$ for a rational number s such that

$$\begin{cases} s \in I_2(r; n) \setminus I_2(r) & \text{if } k \text{ is even;} \\ s \in I_1(r; n) \setminus I_1(r) & \text{if } k \text{ is odd.} \end{cases}$$

Lemma 3.5. *Let $S(r) = (S_1, S_2, S_1, S_2)$ be as in [6, Lemma 3.9]. For simplicity, we write m for m_1 .*

- (1) *If k is even and $[m_1, \dots, m_k, 2n - 2] < s < [m_1, \dots, m_{k-1}]$, then $CS(s)$ contains $(m, S_{1e}, d\langle S_2, S_1 \rangle, S_2, S_{1b}, m)$ as a subsequence, where $1 \leq d \leq 2n - 3$ and $(m + 1, S_{1e}) = (S_{1b}, m + 1) = S_1$.*
- (2) *If k is odd and $[m_1, \dots, m_{k-1}] < s < [m_1, \dots, m_k, 2n - 2]$, then $CS(s)$ contains $(m + 1, S_{2e}, d\langle S_1, S_2 \rangle, S_1, S_{2b}, m + 1)$ as a subsequence, where $1 \leq d \leq 2n - 3$ and $(m, S_{2e}) = (S_{2b}, m) = S_2$.*

Proof. We prove (1) and (2) simultaneously by induction on $k \geq 2$. Let s satisfy

$$\begin{cases} [m_1, \dots, m_k, 2n - 2] < s < [m_1, \dots, m_{k-1}] & \text{if } k \text{ is even;} \\ [m_1, \dots, m_{k-1}] < s < [m_1, \dots, m_k, 2n - 2] & \text{if } k \text{ is odd.} \end{cases}$$

Write s as a continued fraction expansion $s = [l_1, \dots, l_t]$, where $(l_1, \dots, l_t) \in (\mathbb{Z}_+)^t$ and $l_t \geq 2$. Then $l_1 = m$.

Throughout the proof, denote by \tilde{r} and \tilde{s} the rational numbers defined as in [6, Lemma 3.8] for the rational numbers r and s , so that $CS(\tilde{r}) = CT(r)$ and $CS(\tilde{s}) = CT(s)$.

We consider three cases separately.

Case 1. $m_2 = 1$.

In this case, $k \geq 3$, $l_2 = m_2 = 1$ and $t \geq 3$. So we have

$$\tilde{r} = [m_3, \dots, m_k] \quad \text{and} \quad \tilde{s} = [l_3, \dots, l_t].$$

It follows from the assumption that

$$\begin{cases} [m_3, \dots, m_k, 2n-2] < \tilde{s} < [m_3, \dots, m_{k-1}] & \text{if } k \text{ is even;} \\ [m_3, \dots, m_{k-1}] < \tilde{s} < [m_3, \dots, m_k, 2n-2] & \text{if } k \text{ is odd.} \end{cases}$$

This enables us to use the inductive argument. At this point, we divide this case into three subcases.

Case 1.a. $k = 3$.

Since $\tilde{r} = [m_3]$ and $0 < \tilde{s} < [m_3, 2n-2]$, we see by [6, Lemma 5.4] that $CS(\tilde{s})$ contains a subsequence $(m_3 + c, d\langle m_3 \rangle, m_3 + c')$ for some $c, c' \geq 1$ and $0 \leq d \leq 2n-4$. Since $CT(s) = CS(\tilde{s})$, this implies that $CS(s)$ contains a subsequence

$$((m_3 + c)\langle m+1 \rangle, m, d\langle m_3\langle m+1 \rangle, m \rangle, (m_3 + c')\langle m+1 \rangle),$$

where $0 \leq d \leq 2n-4$. In particular, $CS(s)$ contains a subsequence

$$(m+1, m_3\langle m+1 \rangle, m, d\langle m_3\langle m+1 \rangle, m \rangle, m_3\langle m+1 \rangle, m+1).$$

Since $S_1 = (m_3\langle m+1 \rangle)$ and $S_2 = (m)$ by [6, Lemma 3.12(1)], $CS(s)$ contains a subsequence $(m+1, d'\langle S_1, S_2 \rangle, S_1, m+1)$, where $d' = d+1$. This implies the assertion, because $S_2 = (m)$ and $S_{2e} = S_{2b} = \emptyset$.

Case 1.b. $k \geq 4$ is even.

Let $S(\tilde{r}) = (T_1, T_2, T_1, T_2)$ be the decomposition of $S(\tilde{r})$ given by [6, Lemma 3.9]. Then, by the induction hypothesis, $CS(\tilde{s})$ contains $(m_3, T_{1e}, d\langle T_2, T_1 \rangle, T_2, T_{1b}, m_3)$ as a subsequence, where $1 \leq d \leq 2n-3$ and $(m_3+1, T_{1e}) = (T_{1b}, m_3+1) = T_1$. Since $CS(\tilde{s}) = CT(s)$, we see by using [6, Lemma 3.12(2)] that $CS(s)$ contains $(m, S_{1e}, d\langle S_2, S_1 \rangle, S_2, S_{1b}, m)$ as a subsequence, where $1 \leq d \leq 2n-3$ and $(m+1, S_{1e}) = (S_{1b}, m+1) = S_1$. In fact, we have the following identity under the notation of [6, Lemma 3.12(2)]:

$$\begin{aligned} (m, S_{1e}) &= (m, (t_1-1)\langle m+1 \rangle, m, t_2\langle m+1 \rangle, \dots, t_{s_1-1}\langle m+1 \rangle, m, t_{s_1}\langle m+1 \rangle) \\ &= (m, m_3\langle m+1 \rangle, m, t_2\langle m+1 \rangle, \dots, t_{s_1-1}\langle m+1 \rangle, m, t_{s_1}\langle m+1 \rangle). \end{aligned}$$

Thus the “ T -sequence” of (m, S_{1e}) is (m_3, T_{1e}) . Similarly, the “ T -sequence” of (S_{1b}, m) is (T_{1b}, m_3) . By using these facts, we can confirm the assertion above.

Case 1.c. $k \geq 4$ is odd.

Let $S(\tilde{r}) = (T_1, T_2, T_1, T_2)$ be the decomposition of $S(\tilde{r})$ given by [6, Lemma 3.9]. Then, by the induction hypothesis, $CS(\tilde{s})$ contains $(m_3+1, T_{2e}, d\langle T_1, T_2 \rangle, T_1, T_{2b}, m_3+1)$ as a subsequence, where $1 \leq d \leq 2n-3$ and $(m_3, T_{2e}) = (T_{2b}, m_3) = T_2$. Since $CS(\tilde{s}) = CT(s)$, we see by using [6, Lemma 3.12(2)] that $CS(s)$ contains $(m+1, S_{2e}, d\langle S_1, S_2 \rangle, S_1, S_{2b}, m+1)$ as a subsequence, where $1 \leq d \leq 2n-3$ and $(m, S_{2e}) = (S_{2b}, m) = S_2$.

Case 2. $k = 2$ and $m_2 = 2$.

In this case, $r = [m, 2]$ and $[m, 2, 2n-2] < s < [m]$. Then one of the following holds for $s = [l_1, \dots, l_t]$.

- (i) $t \geq 3$, $l_1 = m$, $l_2 = 2$ and $l_3 \leq 2n-3$; or
- (ii) $t \geq 2$, $l_1 = m$ and $l_2 \geq 3$.

If (i) happens, we claim that $CS(s)$ contains a subsequence $(2\langle m \rangle, m+1, d\langle m, m+1 \rangle, 2\langle m \rangle)$, where $0 \leq d \leq 2n-4$. Clearly $\tilde{s} = [1, l_3, \dots, l_t]$. Here, if $l_3 = 1$, then $t \geq 4$ and $CS(\tilde{s}) = CT(s)$ contains a subsequence $(2, 2)$. So $CS(s)$ contains a subsequence $(2\langle m \rangle, m+1, 2\langle m \rangle)$ and therefore the claim holds with $d = 0$. Also if $l_3 \geq 2$, then $CS(\tilde{s}) = CT(s)$ contains a subsequence $(2, (l_3-1)\langle 1 \rangle, 2)$, so that $CS(s)$ contains a subsequence $(2\langle m \rangle, m+1, (l_3-1)\langle m, m+1 \rangle, 2\langle m \rangle)$ and therefore the claim holds with $d = l_3-1 \leq 2n-4$. Then, since $S_1 = (m+1)$ and $S_2 = (m)$, $CS(s)$ contains $(m, d'\langle S_2, S_1 \rangle, S_2, m)$ as a subsequence, where $d' = d+1$. Since $S_1 = (m+1)$, $S_{1e} = S_{1b} = \emptyset$. Hence the assertion holds.

On the other hand, if (ii) happens, we claim that $CS(s)$ contains a subsequence $((l_2-1)\langle m \rangle, m+1, (l_2-1)\langle m \rangle)$, where $l_2-1 \geq 2$. Clearly $\tilde{s} = [l_2-1, l_3, \dots, l_t]$. Here, if either $t = 2$ or $l_3 \geq 2$, then $CS(\tilde{s}) = CT(s)$ contains a subsequence (l_2-1, l_2-1) , so that $CS(s)$ contains a subsequence $((l_2-1)\langle m \rangle, m+1, (l_2-1)\langle m \rangle)$, as desired. Also if $l_3 = 1$, then $t \geq 4$ and $CS(\tilde{s}) = CT(s)$ contains a subsequence (l_2, l_2) . Then $CS(s)$ contains a subsequence $(m, (l_2-1)\langle m \rangle, m+1, (l_2-1)\langle m \rangle, m)$, and therefore $CS(s)$ contains a subsequence $((l_2-1)\langle m \rangle, m+1, (l_2-1)\langle m \rangle)$, as desired. Then, since $S_1 = (m+1)$ and $S_2 = (m)$, $CS(s)$ contains (m, S_2, S_1, S_2, m) as a subsequence, so the assertion holds.

Case 3. Either both $k = 2$ and $m_2 \geq 3$ or both $k \geq 3$ and $m_2 \geq 2$.

In this case, $l_2 \geq 2$. So we have

$$\tilde{r} = [m_2-1, \dots, m_k] \quad \text{and} \quad \tilde{s} = [l_2-1, \dots, l_t].$$

It follows from the assumption that

$$\begin{cases} [m_2 - 1, \dots, m_{k-1}] < \tilde{s} < [m_2 - 1, \dots, m_k, 2n - 2] & \text{if } k \text{ is even;} \\ [m_2 - 1, \dots, m_k, 2n - 2] < \tilde{s} < [m_2 - 1, \dots, m_{k-1}] & \text{if } k \text{ is odd.} \end{cases}$$

This enables us to use the inductive argument. At this point, we divide this case into three subcases.

Case 3.a. $k = 2$ and $m_2 \geq 3$.

Then $\tilde{r} = [m_2 - 1]$ and $0 < \tilde{s} < [m_2 - 1, 2n - 2]$. Hence we see by [6, Lemma 5.4] that $CS(\tilde{s})$ contains a subsequence $(m_2 - 1 + c, d\langle m_2 - 1 \rangle, m_2 - 1 + c')$ for some $c, c' \geq 1$ and $0 \leq d \leq 2n - 4$. Since $CT(s) = CS(\tilde{s})$, this implies that $CS(s)$ contains a subsequence

$$(m, (m_2 - 1)\langle m \rangle, m + 1, d\langle (m_2 - 1)\langle m \rangle, m + 1 \rangle, (m_2 - 1)\langle m \rangle, m),$$

where $0 \leq d \leq 2n - 4$. Since $S_1 = (m + 1)$ and $S_2 = ((m_2 - 1)\langle m \rangle)$ by [6, Lemma 3.12(3)], $CS(s)$ contains a subsequence $(m, d'\langle S_2, S_1 \rangle, S_2, m)$, where $d' = d + 1$. Since $S_1 = (m + 1)$ and therefore $S_{1e} = S_{1b} = \emptyset$, the assertion holds.

Case 3.b. $k \geq 3$ is even and $m_2 \geq 2$.

Let $S(\tilde{r}) = (T_1, T_2, T_1, T_2)$ be the decomposition of $S(\tilde{r})$ given by [6, Lemma 3.9]. Then, by the inductive hypothesis, $CS(\tilde{s})$ contains $(m_2, T_{2e}, d\langle T_1, T_2 \rangle, T_1, T_{2b}, m_2)$ as a subsequence, where $1 \leq d \leq 2n - 3$ and $(m_2 - 1, T_{2e}) = (T_{2b}, m_2 - 1) = T_2$. Since $CS(\tilde{s}) = CT(s)$, we see by using [6, Lemma 3.12(4)] that $CS(s)$ contains $(m, S_{1e}, d\langle S_2, S_1 \rangle, S_2, S_{1b}, m)$ as a subsequence, where $1 \leq d \leq 2n - 3$ and $(m + 1, S_{1e}) = (S_{1b}, m + 1) = S_1$.

Case 3.c. $k \geq 3$ is odd and $m_2 \geq 2$.

Let $S(\tilde{r}) = (T_1, T_2, T_1, T_2)$ be the decomposition of $S(\tilde{r})$ given by [6, Lemma 3.9]. Then, by the inductive hypothesis, $CS(\tilde{s})$ contains $(m_2 - 1, T_{1e}, d\langle T_2, T_1 \rangle, T_2, T_{1b}, m_2 - 1)$ as a subsequence, where $1 \leq d \leq 2n - 3$ and $(m_2, T_{1e}) = (T_{1b}, m_2) = T_1$. Since $CS(\tilde{s}) = CT(s)$, we see by using [6, Lemma 3.12(4)] that $CS(s)$ contains $(m + 1, S_{2e}, d\langle S_1, S_2 \rangle, S_1, S_{2b}, m + 1)$ as a subsequence, where $1 \leq d \leq 2n - 3$ and $(m, S_{2e}) = (S_{2b}, m) = S_2$.

The proof of Lemma 3.5 is now completed. \square

4. PROOF OF MAIN THEOREM 2.2(1)

By Remark 2.3(2), we may let $r = [m_1, \dots, m_k]$ with $m_1 = m \geq 2$ and $k \geq 2$. Also let s and s' be distinct rational numbers in $I_1(r; n) \cup I_2(r; n)$. Suppose on the contrary that the simple loops α_s and $\alpha_{s'}$ are homotopic in $\mathcal{S}(r; n)$, i.e., u_s and $u_{s'}^{\pm 1}$ are conjugate in $G(r; n)$. By [6, Lemma 4.11], there is a reduced nontrivial annular diagram M over $G(r; n) = \langle a, b \mid u_r^n \rangle$ with $(\phi(\alpha)) = (u_s)$ and $(\phi(\delta)) = (u_{s'}^{\pm 1})$, where α and δ are, respectively, outer and inner boundary cycles of M . Since $s, s' \in I_1(r; n) \cup I_2(r; n)$, we see by Lemma 3.1 that $CS(\phi(\alpha))$ and $CS(\phi(\delta))$ do not contain $((2n-1)\langle S_1, S_2 \rangle)$ nor $((2n-1)\langle S_2, S_1 \rangle)$ as a subsequence. So by [6, Corollary 4.17], M is shaped as in [6, Figure 3(a)] or [6, Figure 3(b)].

Lemma 4.1. *M is shaped as in [6, Figure 3(a)], that is, M satisfies the conclusion of [6, Corollary 4.17(1)].*

Proof. Suppose on the contrary that M is shaped as in [6, Figure 3(b)]. Then $(\phi(\alpha)) = (u_s)$ contains a subword of the cyclic word $(u_r^{\pm n})$ which is a product of $4n-2$ pieces, but is not a product of less than $4n-2$ pieces (see [6, Convention 4.7(3) and Theorem 4.15(4)]). Since $4n-2 \geq 6$, this together with [6, Lemma 4.2(2c)] implies that $CS(\phi(\alpha)) = CS(s)$ contains both S_1 and S_2 as subsequences. So by Lemma 3.3 $s \notin I_1(r) \cup I_2(r)$. Then by Lemmas 3.4 and 3.5, (u_s) contains a subword w for which $S(w)$ is a subsequence of $CS(s)$ such that

$$S(w) = \begin{cases} (m+1, S_{2e}, S_1, S_2, S_1, S_{2b}, m+1) & \text{if } k \text{ is even and } s \in I_1(r; n); \\ (m, S_{1e}, d\langle S_2, S_1 \rangle, S_2, S_{1b}, m) & \text{if } k \text{ is even and } s \in I_2(r; n); \\ (m, S_{1e}, S_2, S_1, S_2, S_{1b}, m) & \text{if } k \text{ is odd and } s \in I_2(r; n); \\ (m+1, S_{2e}, d\langle S_1, S_2 \rangle, S_1, S_{2b}, m+1) & \text{if } k \text{ is odd and } s \in I_1(r; n), \end{cases}$$

where $1 \leq d \leq 2n-3$, $(m+1, S_{1e}) = (S_{1b}, m+1) = S_1$ and $(m, S_{2e}) = (S_{2b}, m) = S_2$. In any of the above four cases, we see by arguing as in the proof of [6, Lemma 6.1] that there is a face D in the outer boundary layer of M such that $\phi(\partial D^+)$ is a subword of w . Since $CS(\phi(\partial D)) = ((2n)\langle S_1, S_2 \rangle)$, this implies that $S(\phi(\partial D^-))$ must contain (S_1, S_2, ℓ) as a subsequence for some $\ell \in \mathbb{Z}_+$. In more detail, if $S(w)$ is of the first or fourth form, then since $\phi(\partial D^+)$ is a subword of w , $S(\phi(\partial D^-))$ must contain (S_1, S_2, S_1) as a subsequence. On the other hand, if $S(w)$ is of the second or third form, then $S(\phi(\partial D^-))$ must contain $(\ell_1, S_2, S_1, S_2, \ell_2)$ as a subsequence for some $\ell_1, \ell_2 \in \mathbb{Z}_+$. But then by [6, Lemma 4.2(2)], the word $\phi(\partial D^-)$ cannot be expressed as a product of 2 pieces of $(u_r^{\pm 1})$, contradicting [6, Figure 3(b)] (cf. [6, Corollary 4.17(2)]). \square

Lemma 4.2. $s, s' \notin I_1(r) \cup I_2(r)$.

Proof. Suppose on the contrary that s or s' lies in $I_1(r) \cup I_2(r)$. Without loss of generality, assume that $s \in I_1(r) \cup I_2(r)$. By Lemma 3.3, either S_1 or S_2 does not occur in $CS(s)$ as a subsequence. But then by the feature of [6, Figure 3(a)], $CS(\phi(\delta)) = CS(s')$ contains both S_1 and S_2 as subsequences, which implies by Lemma 3.3 that $s' \notin I_1(r) \cup I_2(r)$ and therefore $s' \in (I_1(r; n) \setminus I_1(r)) \cup (I_2(r; n) \setminus I_2(r))$. Then by the same argument as in the proof of Lemma 4.1, we see that $CS(s) = CS(\phi(\alpha))$ contains (S_1, S_2, ℓ) as a subsequence for some $\ell \in \mathbb{Z}_+$, a contradiction. \square

Lemma 4.3. Both $CS(s)$ and $CS(s')$ consist of m and $m + 1$.

Proof. By Lemma 4.2 together with Lemmas 3.4 and 3.5, $CS(s)$ and $CS(s')$ contain both S_1 and S_2 . Hence by [6, Lemmas 3.5 and 3.9], both $CS(s)$ and $CS(s')$ consist of m and $m + 1$. \square

At this point, we introduce the concept for a vertex of M to be converging, diverging or mixing (cf. [5, Section 7]). To this end, we subdivide the edges of M so that the label of any oriented edge in the subdivision has length 1. We call each of the edges in the subdivision a *unit segment* in order to distinguish them from the edges in the original M .

Definition 4.4. (1) A vertex x in M is said to be *converging* (resp., *diverging*) if the set of labels of incoming unit segments of x is $\{a, b\}$ (resp., $\{a^{-1}, b^{-1}\}$). See Figure 1 and its caption for description.

(2) A vertex x in M is said to be *mixing* if the set of labels of incoming unit segments of x is $\{a, a^{-1}, b, b^{-1}\}$. See Figure 2 and its caption for description.

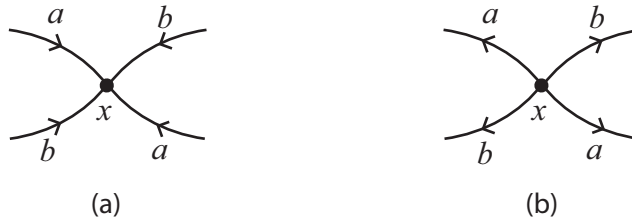


FIGURE 1. Orient each of the unit segment so that the associated label is equal to a or b . Then a vertex x is (a) converging (resp., (b) diverging) if all unit segments incident on x are oriented so that they are converging into x (resp., diverging from x).



FIGURE 2. A vertex x is mixing if it looks like as in the above when we orient the segments as in [6, Convention 6.7], namely, the change of directions of consecutive arrowheads represents the change from positive (negative, resp.) words to negative (positive, resp.) words.

The proof of the following lemma is a slight modification of that of [5, Proposition 7.5(3)].

Lemma 4.5. *We may assume that every vertex x of M with degree 4 is either converging or diverging. To be precise, we can modify the reduced nontrivial annular diagram M into a reduced nontrivial annular diagram M' keeping the outer and inner boundary labels unchanged so that every vertex of M' with degree 4 is either converging or diverging.*

Proof. Suppose on the contrary that there is a vertex $x \in M$ with degree 4 such that x is neither converging nor diverging. We may assume x is the vertex between D_1 and D_2 . Then x has one of the five types as depicted in Figure 3, where c_i and d_i ($i = 1, 2$) are positive integers, up to simultaneous reversal of the edge orientations and up to the reflection in the vertical edge passing through the vertex x . To see this, let L be the set of labels of incoming unit segments of x , and orient each of the unit segment so that the associated label is equal to a or b as in Figure 1. If $L = \{a^{\pm 1}, b^{\pm 1}\}$, then we obtain the situation (a) or (b) in Figure 3 which follows [6, Convention 6.7]. If L consists of three elements, then we may assume that a and a^{-1} , respectively, appear as the label of the upper left and lower right incoming unit segments and that b or b^{-1} does not belong to L . Then we obtain the situation (c) or (d) in Figure 3. If L consists of two elements, then we may assume both the upper left and lower right incoming unit segments have label a , and both the upper left and lower right incoming unit segments have label b^{-1} , because x is not converging nor diverging. In this case, we have the situation (e) in Figure 3.

Assume that x is as depicted in Figure 3(a). Then, for each $i = 1, 2$, c_i is a term of $CS(\phi(\partial D_i)) = CS(u_r^n)$ and so is equal to m or $m + 1$. Hence the term,

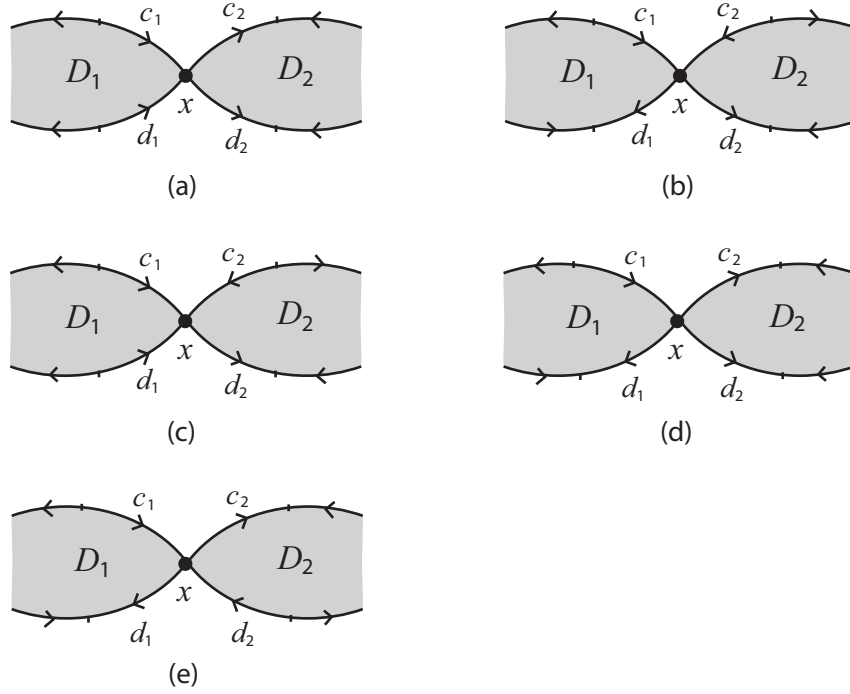


FIGURE 3. The five possible types of a vertex $x \in M$ with degree 4 such that x is neither converging nor diverging

$c_1 + c_2$, of $CS(\phi(\alpha)) = CS(s)$ is at least $2m$. By Lemma 4.3, $CS(s)$ consists of m and $m + 1$. But since $2m > m + 1$, we obtain a contradiction.

Assume that x is as depicted in Figure 3(b). Then (c_1, c_2) is a subsequence of $CS(\phi(\alpha)) = CS(s)$. Since $CS(\phi(\alpha)) = CS(s)$ consists of m and $m + 1$ by Lemma 4.3, the only possibility is that $c_1 = c_2 = m$ and $d_1 = d_2 = 1$. But then there is term 1 in $CS(s')$, which is a contradiction, because $CS(s')$ also consists of m and $m + 1$ again by Lemma 4.3.

Assume that x is as depicted in Figure 3(c). Then (c_1, c_2) is a subsequence of $CS(\phi(\alpha)) = CS(s)$ and $d_1 + d_2$ is a term of $CS(\phi(\delta)) = CS(s')$. Thus each of c_1 , c_2 and $d_1 + d_2$ is either m or $m + 1$ by Lemma 4.3. Moreover, $c_2 + d_2$ is a term of $CS(u_r^n)$ and hence it is either m or $m + 1$. So, we have the following two possibilities:

- (i) $c_1 = m$, $c_2 = m$, $d_1 = m$, $d_2 = 1$;
- (ii) $c_1 = m + 1$, $c_2 = m$, $d_1 = m$, $d_2 = 1$.

In either case, since $c_2 = d_1$, we can transform M so that x is diverging as in Figure 4. To be precise, we cut M at the black vertex in the left figure in Figure 4 and then identify the two white vertices. The resulting diagram is illustrated in the right figure in Figure 4, where the black vertex is the image of the white vertices. It should be noted that this modification does not change the boundary labels of M and the new vertex of M is converging or diverging.

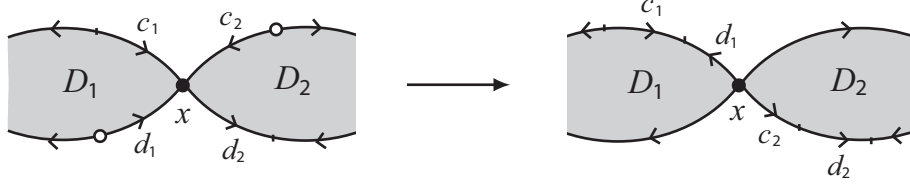


FIGURE 4. The transformation of Figure 3(c) when $c_2 = d_1$ so that x is diverging

Assume that x is as depicted in Figure 3(d). Then $c_1 + c_2$ is a term of $CS(\phi(\alpha)) = CS(s)$ and (d_1, d_2) is a subsequence of $CS(\phi(\delta)) = CS(s')$. Thus each of $c_1 + c_2$, d_1 and d_2 is either m or $m + 1$ by Lemma 4.3. Moreover, $c_1 + d_1$ is a term of $CS(u_r^n)$ and hence it is either m or $m + 1$. So, we have the following two possibilities:

- (i) $c_1 = 1, c_2 = m, d_1 = m, d_2 = m$;
- (ii) $c_1 = 1, c_2 = m, d_1 = m, d_2 = m + 1$.

In either case, since $c_2 = d_1$, we can transform M so that x is converging as in Figure 5.

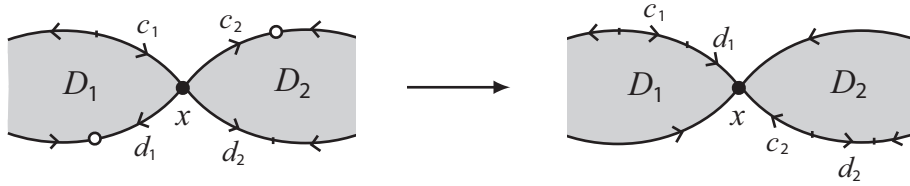


FIGURE 5. The transformation of Figure 3(d) when $c_2 = d_1$ so that x is converging

Assume that x is as depicted in Figure 3(e). Then $c_1 + c_2$ is a term of $CS(\phi(\alpha)) = CS(s)$ and $d_1 + d_2$ is a term of $CS(\phi(\delta)) = CS(s')$. Thus each of $c_1 + c_2$ and $d_1 + d_2$ is either m or $m + 1$ by Lemma 4.3. Moreover, for each $i = 1, 2$, $c_i + d_i$ is a term of $CS(u_r^n)$ and hence it is either m or $m + 1$. So, we have the following six possibilities:

- (i) $c_1 + c_2 = m, d_1 + d_2 = m, c_1 + d_1 = m, c_2 + d_2 = m$;
- (ii) $c_1 + c_2 = m, d_1 + d_2 = m + 1, c_1 + d_1 = m, c_2 + d_2 = m + 1$;
- (iii) $c_1 + c_2 = m, d_1 + d_2 = m + 1, c_1 + d_1 = m + 1, c_2 + d_2 = m$;
- (iv) $c_1 + c_2 = m + 1, d_1 + d_2 = m, c_1 + d_1 = m, c_2 + d_2 = m + 1$;
- (v) $c_1 + c_2 = m + 1, d_1 + d_2 = m, c_1 + d_1 = m + 1, c_2 + d_2 = m$;
- (vi) $c_1 + c_2 = m + 1, d_1 + d_2 = m + 1, c_1 + d_1 = m + 1, c_2 + d_2 = m + 1$.

If (i), (ii), (v) or (vi) happens, then $c_2 = d_1$. Thus, as illustrated in Figure 5, we may transform M so that x is converging. If (iii) or (vi) happens, then $c_1 = d_2$. So we can transform M so that x is diverging as in Figure 4. \square

Lemma 4.6. $r \neq [m, 2]$.

Proof. Suppose on the contrary that $r = [m, 2]$. By [6, Lemma 3.12(3)], $S_1 = (m + 1)$ and $S_2 = (m)$.

Claim. $s, s' \notin I_1(r; n) \setminus I_1(r)$.

Proof of Claim. Suppose on the contrary that s or s' is contained in $I_1(r; n) \setminus I_1(r)$. Without loss of generality, assume that $s \in I_1(r; n) \setminus I_1(r)$, i.e., $[m + 1] < s \leq [m, 1, 2]$. Here, if $s = [m, 1, 2]$, then $CS(s) = ((m + 1, m + 1, m, m + 1, m + 1, m))$. Clearly $CS(\phi(\partial D)) = ((2n \langle m + 1, m \rangle))$ for every face D in M . Then we see by Lemma 4.5 that there are two successive 2-cells, say D_1 and D_2 , in M such that $S(\phi(\partial D_1^+)) = (\dots, m + 1)$, $S(\phi(\partial D_2^+)) = (m + 1, \dots)$ and such that $S(\phi(\partial D_1^+ \partial D_2^+)) = (\dots, m + 1, m + 1, \dots)$. By Lemma 3.1(1), neither $S(\phi(\partial D_1^-))$ nor $S(\phi(\partial D_2^-))$ can contain $((2n - 2) \langle m + 1, m \rangle, m + 1)$ as a subsequence, and therefore we must have $S(\phi(\partial D_1^+)) = (m + 1, m, m + 1)$ and $S(\phi(\partial D_2^+)) = (m + 1, m, m + 1)$. It then follows that M consists of 2-cells D_1 and D_2 with $CS(\phi(\delta^{-1})) = CS(\phi(\partial D_1^- \partial D_2^-)) = (((2n - 2) \langle m, m + 1 \rangle, m, (2n - 2) \langle m, m + 1 \rangle, m))$. Since $CS(s') = CS(\phi(\delta^{-1}))$, this implies that $s' = [m, 2, 2n - 2]$. But then s' is not contained in $I_1(r; n) \cup I_2(r; n)$, a contradiction.

So assume that $[m + 1] < s < [m, 1, 2]$. Write s as a continued fraction expansion $s = [l_1, l_2, \dots, l_t]$, where $t \geq 1$ and $(l_1, \dots, l_t) \in (\mathbb{Z}_+)^t$ and $l_t \geq 2$. Then $l_1 = m, l_2 = 1$ and either $l_3 \geq 3$ or both $l_3 \geq 2$ and $t \geq 4$. In either case, $CS(\phi(\alpha)) = CS(s)$ contains $(m + 1, m + 1, m + 1)$ as a subsequence. Then we see by Lemma 4.5 that there are three successive faces D_1, D_2, D_3 in M such that $S(\phi(\partial D_1^+)) = (\dots, m + 1)$, $S(\phi(\partial D_2^+)) = (m + 1)$, $S(\phi(\partial D_3^+)) = (m + 1, \dots)$ and such that $S(\phi(\partial D_1^+ \partial D_2^+ \partial D_3^+)) = (\dots, m + 1, m + 1, m + 1, \dots)$. But then, since $CS(\phi(\partial D_2)) = ((2n \langle m + 1, m \rangle))$, $S(\phi(\partial D_2^-)) = (m, (2n - 1) \langle m + 1, m \rangle)$. Hence $CS(\phi(\delta)) = CS(s')$ contains $((2n - 2) \langle m + 1, m \rangle, m + 1)$ as a subsequence, which is a contradiction to Lemma 3.1(1). \square

By the above claim, we have $s \in I_2(r; n) \setminus I_2(r)$. Then by Lemma 3.5(1), $CS(s)$ contains $(m, d\langle m, m+1 \rangle, m, m)$ where $1 \leq d \leq 2n-3$, because $S_{1e} = S_{1b} = \emptyset$. Again by Lemma 4.5, we see that there are two successive 2-cells, say D_1 and D_2 , in M such that $S(\phi(\partial D_1^+)) = (\cdots, m)$, $S(\phi(\partial D_2^+)) = (m, \cdots)$ and such that $S(\phi(\partial D_1^+ \partial D_2^+)) = (\cdots, m, m, \cdots)$. Then since $CS(\phi(\partial D_1)) = CS(\phi(\partial D_2)) = ((2n\langle m+1, m \rangle))$, $S(\phi(\partial D_1^-)) = (\cdots, m+1)$, $S(\phi(\partial D_2^-)) = (m+1, \cdots)$ and $S(\phi(\partial D_1^- \partial D_2^-)) = (\cdots, m+1, m+1, \cdots)$. This implies that $CS(\phi(\delta)) = CS(s')$ contains two consecutive terms $(m+1, m+1)$. But then $s' \notin I_2(r; n) \setminus I_2(r)$, contrary to the above claim. \square

By \tilde{r} , \tilde{s} and \tilde{s}' , we denote the rational numbers defined as in [6, Lemma 3.8] for the rational numbers r, s and s' so that $CS(\tilde{r}) = CT(r)$, $CS(\tilde{s}) = CT(s)$ and $CS(\tilde{s}') = CT(s')$.

Lemma 4.7. $\tilde{s}, \tilde{s}' \in I_1(\tilde{r}; n) \cup I_2(\tilde{r}; n)$.

Proof. Write s, s' as continued fraction expansions $s = [p_1, p_2, \dots, p_h]$ and $s' = [q_1, q_2, \dots, q_l]$, where $p_i, q_j \in \mathbb{Z}_+$ and $p_h, q_l \geq 2$. Since both $CS(s)$ and $CS(s')$ consist of m and $m+1$ by Lemma 4.3, we have $p_1 = q_1 = m$ by [6, Lemma 3.5]. If $m_2 = 1$, then by [6, Corollary 3.14(1)], $(m+1, m+1)$ appears in S_1 , so in $CS(s)$ and $CS(s')$. This implies by [6, Lemma 3.5] that $p_2 = q_2 = 1$. Also if $m_2 \geq 2$, then by [6, Corollary 3.14(1)] together with Lemma 4.6, (m, m) appears in S_2 , so in $CS(s)$ and $CS(s')$. This implies by [6, Lemma 3.5] that $p_2, q_2 \geq 2$.

Therefore, by [6, Lemma 3.8], if $\tilde{r} = [m_3, \dots, m_k]$, then $\tilde{s} = [p_3, \dots, p_h]$ and $\tilde{s}' = [q_3, \dots, q_l]$. This together with the fact $p_1 = q_1 = m$ and $p_2 = q_2 = 1$ yields the assertion. Also, by [6, Lemma 3.8], if $\tilde{r} = [m_2 - 1, m_3, \dots, m_k]$, then $\tilde{s} = [p_2 - 1, p_3, \dots, p_h]$ and $\tilde{s}' = [q_2 - 1, q_3, \dots, q_l]$. This together with the fact $p_1 = q_1 = m$ yields the assertion. \square

Lemma 4.8. *The unoriented loops $\alpha_{\tilde{s}}$ and $\alpha_{\tilde{s}'}$ represent the same conjugacy class in $G(\tilde{r}; n)$.*

Proof. Let \tilde{R} be the symmetrized subset of $F(a, b)$ generated by the single relator $u_{\tilde{r}}^n$ of the upper presentation $G(\tilde{r}; n) = \langle a, b \mid u_{\tilde{r}}^n \rangle$. Due to Lemma 4.5, we can construct, as described in [5, Section 8], a reduced nontrivial annular \tilde{R} -diagram \tilde{M} from a given R -diagram M such that $u_{\tilde{s}}$ is an outer boundary label and $u_{\tilde{s}'}^{\pm 1}$ is an inner boundary label of \tilde{M} . By [6, Lemma 4.11], this proves that the unoriented loops $\alpha_{\tilde{s}}$ and $\alpha_{\tilde{s}'}$ represent the same conjugacy class in $G(\tilde{r}; n)$. \square

Remark 4.9. In the statements of Lemmas 4.7 and 4.8, we may assume that $0 < \tilde{r} \leq 1/2$. The reason is as follows. Note that there is a homeomorphism $f : (S^3, K(\tilde{r})) \rightarrow (S^3, K(1 - \tilde{r}))$ preserving the bridge sphere such that $f(\alpha_{\tilde{s}}) = \alpha_{1-\tilde{s}}$ and $f(\alpha_{\tilde{s}'}) = \alpha_{1-\tilde{s}'}$ and that f induces an isomorphism from $G(\tilde{r}; n)$ to $G(1 - \tilde{r}; n)$ sending the standard generators a and b to a and b^{-1} , respectively. In fact, such a homeomorphism is obtained as the composition of the natural homeomorphisms

$$(S^3, K(\tilde{r})) \rightarrow (S^3, K(-\tilde{r})) \rightarrow (S^3, K(1 - \tilde{r})),$$

where the latter homeomorphism is explained in [2, the end of Section 3]. Moreover, the conjugacy diagram over $G(\tilde{r}; n)$ is obtained as the isomorphic image of the conjugacy diagram over $G(1 - \tilde{r}; n)$. Thus, if $1/2 < \tilde{r} < 1$, then we take $1 - \tilde{r}$ instead of \tilde{r} . Here, it should be noted that if $\tilde{r} = [1, n_2, \dots, n_h]$, then $1 - \tilde{r} = [1 + n_2, n_3, \dots, n_h]$ and that the rational numbers $1 - \tilde{s}$ and $1 - \tilde{s}'$ are contained in $I_1(1 - \tilde{r}; n) \cup I_2(1 - \tilde{r}; n)$.

By applying Lemmas 4.7 and 4.8 (and Remark 4.9 if necessary) inductively, we finally arrive at the situation that for either $r' = 1/p$ or $r' = [p, 2]$ for some integer $p \geq 2$, there are two rational numbers $t, t' \in I_1(r'; n) \cup I_2(r'; n)$ for which the simple loops α_t and $\alpha_{t'}$ represent the same conjugacy class in $G(r'; n)$. The former is a contradiction to [6, Main Theorem 2.5(1)], and the latter is a contradiction to Lemma 4.6. This completes the proof of Main Theorem 2.2(1). \square

5. PROOF OF MAIN THEOREM 2.2(3)

Let $S(r) = (S_1, S_2, S_1, S_2)$ be as in [6, Lemma 3.9]. We recall the following lemma.

Lemma 5.1. (1) ([2, Theorem 6.3]) *Suppose that v is a cyclically alternating word which represents the trivial element in $G(K(r)) = \langle a, b \mid u_r \rangle$. Then the cyclic word (v) contains a subword w of the cyclic word $(u_r^{\pm 1})$ such that $S(w)$ is (S_1, S_2, ℓ) or (ℓ, S_2, S_1) for some $\ell \in \mathbb{Z}_+$.*

(2) ([3, Corollary 4.12]) *Suppose that v is a cyclically alternating word which represents the trivial element in $G(r; n) = \langle a, b \mid u_r^n \rangle$. Then the cyclic word (v) contains a subword w of the cyclic word $(u_r^{\pm n})$ such that $S(w)$ is $((2n - 1)\langle S_1, S_2 \rangle, \ell)$ or $(\ell, (2n - 1)\langle S_2, S_1 \rangle)$, where $\ell \in \mathbb{Z}_+$.*

Suppose on the contrary that there is a rational number s in $I_1(r; n) \cup I_2(r; n)$ for which $u_s^t = 1$ in $G(r; n)$ for some integer $t \geq 1$. Then clearly $u_s^t = 1$ also in $G(K(r))$. Since $G(K(r))$ is torsion-free, $u_s = 1$ in $G(K(r))$. By [2,

Main Theorem 2.3], this implies that s lies in the $\hat{\Gamma}_r$ -orbit of r or ∞ . Hence $|u_s| > |u_r|$.

On the other hand, since $u_s^t = 1$ in $G(r; n)$, Lemma 5.1(2) implies that we may write $\bar{u}_s^t \equiv wz$, where \bar{u}_s is a cyclic permutation of u_s and w is a subword of (u_s^t) as described in Lemma 5.1(2). Since $s \in I_1(r; n) \cup I_2(r; n)$, w cannot be contained in \bar{u}_s by Lemma 3.1. This yields that \bar{u}_s is a proper initial subword of w . Then \bar{u}_s is a subword of the cyclic word $(u_r^{\pm n})$. Here, since $|\bar{u}_s| = |u_s| > |u_r|$, we may put $\bar{u}_s \equiv v^d v_1$, where $d \in \mathbb{Z}_+$, v is a cyclic permutation of u_r or u_r^{-1} and $|v_1| < |u_r|$. Note that $|v_1| \geq 1$, for otherwise we would have $\bar{u}_s \equiv v^d$ so that $CS(s) = ((2d\langle S_1, S_2 \rangle))$, which yields that $d \geq 2$ since $s \neq r$ and that $s = dq/dp$ if $r = q/p$ by [6, Remark 3.11], a contradiction.

Then $v_1 = \bar{u}_s = 1$ in $G(K(r))$. Moreover v_1 is a proper beginning subword of v and so a proper beginning subword of \bar{u}_s . This implies that v_1 is cyclically alternating, because \bar{u}_s is cyclically alternating and $|v_1| = |u_s| - d|u_r|$ is even. Also since $v_1 = 1$ in $G(K(r))$, Lemma 5.1(1) implies that $CS(v_1)$ contains (S_1, S_2) or (S_2, S_1) as a proper subsequence. So $|v_1| > \frac{1}{2}|u_r| = \frac{1}{2}|v|$. Now, let v_2 be the terminal subword of v such that $v \equiv v_1 v_2$. Then v_2 is also cyclically alternating and $v_2 = 1$ in $G(K(r))$. Thus the above argument implies that $|v_2| > \frac{1}{2}|v|$. This contradicts the inequality $|v_2| = |v| - |v_1| < \frac{1}{2}|v|$. \square

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